Competition #10

The Junior Online Math Olympiad

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Short Questions

1. Suppose a, b, c are positive real numbers such that $a^2 + b^2 + c^2 = 4$ and $4(a^2 + 2) = (a^2 + b + c)^2$. The maximum possible value of $a + b + c$ can be written as $\sqrt{x} + y$, where x and y are positive integers. Find the value of $x + y$.

2. In the Faugauss Bank, the security code for their clients have four digits, along with a special control number. It is defined as the remainder when the four-digit security code, elevated to the eleventh power, is divided by seven. For instance, a valid security code would be 1234-4. If Hilbert's security code begins with 2014, what is its final number?

3. In a class of 20 students, a test of 10 questions was conducted, in which for each right answer, 1 mark was given and for wrong answer 4 marks were deducted and a zero was given for not attempting a question. In that test, Cody, Yan Yau and Aditya topped with 10 marks each (they are the only toppers). If nobody in the class is bad enough to get a score less than 0, then the number of ways in which the total marks of the class can be 143 is N. What is the digit sum of N ?

4. Suppose p, $p^2 + 2$, $p^3 + 2$, $p^4 + 2$ are primes. Find the sum of all possible values of p.

5. Evaluate the smallest value that the expression:

$$\left(\frac{2 + 3y^2 + 3x^2(1 + y^2)^2 + x(6y + 6y^3)}{y + x(1 + y^2)}\right)^2$$

can attain, if x, y are positive numbers.

6. Find the sum of the absolute value of all integers n such that $|n(n^2 - n + 1) - 6|$ is a prime.

7. Let $a_1, a_2, ...$ be an infinite sequence of real numbers such that $a_1 = \frac{1}{2014}$ and $(a_n + 1)(a_{n-1} - 1) = -1$ for all positive integers $n \geq 2$. Determine the value of a_{2014}.

8. Let ABC be a triangle with area 4, $\angle ABC = 30°$ and $AB \cdot BC \cdot CA = 48$. It's perimeter can be written as $\sqrt{a\sqrt{b} + c} + d$ where b is not divisible by the square of a prime and a, b, c, d are positive integers. Find $a+b+c+d$.

9. Aditya is playing a game named "Cube It". He starts with the number 1 on the screen and at Level 1. For all positive integers n, when Aditya reaches level n, he can press a button that will increase the number on the screen by n. Also, there's another button that he can press to cube root the number on the screen. If it's a perfect cube, then Aditya's level increases by 1, 2, ..., or 1000 randomly each with equal probability. Or else, he lose. Aditya wins the game if his level is greater than 2014. Given that Aditya plays optimally, the probability that he'll win the game is $\frac{a}{b}$, where a and b are coprime positive integers. Find $a + b$.

10. (ZS) Let ABC be a triangle such that $AC = 2(BC - AB)$. Let D be the point of tangency of the incircle of ABC to AC. Find $\frac{AC}{AD}$.

Long Questions

Explain your answer for each question.

1. Show that $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq \frac{9}{a+b+c}$, for all positive values of a, b, c

2. a, b and c are the roots of the equation $x^3 - x^2 + 2x - 3 = 0$. Display an equation with integer coefficients such that its roots are $a - \frac{1}{b+c-1}$, $b - \frac{1}{a+c-1}$ and $c - \frac{1}{a+b-1}$.

3. Prove that there are infinitely many triplets of positive integers (a, b, c) such that:
$$a^3 - b(b+1) = (a-2)(c^2+1) + 2$$