Competition #11

The Junior Online Math Olympiad

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Short Questions

1. (Melodies) What is the number of real solutions for the equation
$$(x^{10660}+1)(1+x^2+x^4+\ldots+x^{10658}) = 10660x^{10659}?$$

2. (Adi) Given that $f(x)$ is a cubic polynomial with real coefficients and $f(1) = 3, f(3) = 8, f(8) = 13, f(13) = 19$. Then $f(26) = \frac{a}{b}$ such that $a, b \in \mathbb{N}$ and $\gcd(a, b) = 1$. Find $a + b$.

3. (Adi) There is a cubic die with 6 faces numbered from 1 to 6. Due to the composition of the die, it's not fair one. Probability of getting a number n on the top is ' $n\times$ probability of getting 1 ' . Then find the probability that the sum of numbers obtained on top from 3 throws of the die, is 15.

4. (Guilherme) Evaluate the smallest value that the expression $\left(\frac{2+3y^2+3x^2(1+y^2)^2+x(6y+6y^3)}{y+x(1+y^2)}\right)^2$ can attain, if x, y are positive numbers.

5. (Guilherme) The number 1 is a double root in the equation $x^4 - 2x^3 - 3x^2 + ax + b = 0$. Evaluate the digit sum of b^a.

6. (Guilherme) Consider the triangle ABC on the cartesian plane, with vertices $A = (0; 0)$, $B = (3; 4)$, $C = (8; 0)$. The rectangle $MNPQ$ has its vertices M, N on the x-axis, and the vertices Q and P are on the sides $\overline{AB}$ and $\overline{BC}$, respectively. Among all rectangles built this way, the one that has maximum area is the one in which the point P is located at $(p_x; p_y)$. Evaluate $p_x \cdot p_y$.

7. (ZS) Let $ABCD$ be a quadrilateral such that it's diagonals intersect at E. $AE = 1, BE = 2, AD = 4, CE = 8$. DE has integer length. $AB \cdot CD = 25$. Find the value of $100BC$.

8. (ZS) Let ABC be a triangle such that $AC = 2BC$. The angle bisector of $\angle ACB$ meets AB at F. E is a point on AC such that $AE = 3EC$. Let BE intersect CF at G. AG intersects BC at D. Find $\frac{AG}{GD}$.

9. Let $f(x) = \frac{x^2}{2x^2-2x+1}$. Determine the value of $f(\frac{1}{2015}) + f(\frac{2}{2015}) + ... + f(\frac{2015}{2015})$.

10. (ZS) Let ABC be a triangle and let Y be on ray CB such that $CB = 2YB$. Let X be on ray AY such that $XY = AY$. Suppose $AC = 2BX$. Find, in degrees, $\angle CTA$, where T is the midpoint of BX.

Long Questions

Explain your answer for each question.

1. (ZS) Prove that for all positive real numbers a, b, c, such that $a^2+b^2+c^2 = 2(ab+bc+ca)$, $\sum_{cyc} \frac{(a+b)^2}{b^2+c^2} \geq 4$.

   Solution

   By Titu's lemma,

   $$\sum_{cyc} \frac{(a+b)^2}{b^2+c^2} \geq \frac{4(a+b+c)^2}{2(a^2+b^2+c^2)} = 2 + \frac{4(ab+bc+ca)}{a^2+b^2+c^2} = 4.$$

2. (Guilherme) Define $M(x)$ as the monic polynomial with integer coefficients of least degree such that $M(x) = 0$. For instance, $M(3) = x-3$, $M(\sqrt{2}) = x^2-2$ and $M(2-\sqrt{2}) = x^2-4x+2$. Find, with proof, $M(\sqrt[3]{3}-\sqrt[3]{2})$.

3. (Guilherme) Show that, for some positive integer values of $(a_1, a_2, \cdots, a_n, b_n)$, the condition $\sum_{i=1}^{n} a_i = {b_n}^2$ can always be attained for integer values of $n \geq 2$.