



Competition #12

The Junior Online Math Olympiad

Aditya, Guilherme Dela Corte, Melodies Sim, Navi Nahc

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Short Questions

1. (Aditya) Given that

$$a + \frac{1}{b + \frac{1}{c + \frac{1}{d + \frac{1}{e}}}} = \frac{101061}{14254}$$

and all of a, b, c, d, e are **positive integers**, then find $a + b + c + d + e$.

2. (Navi) Let positive reals a, b, c satisfy $abc = 1$. Find the last 3 digits of the minimum value of

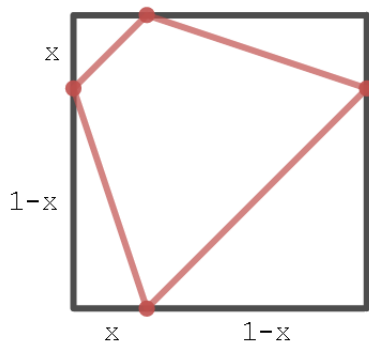
$$(a + 2015b)(b + 2015c)(c + 2015a)$$

3. (Navi) Let N be a square-free integer, with divisors $1 < d_2 < \dots < d_8$. Given that $d_4 = 2d_3$ and $d_7 - d_6 = 901$, find N .
4. (Navi) Let ABC be a triangle and let two points P, Q lie on segment BC so that Q is closer to B than P . Let circumcircle of APC intersect line AB at D and intersect AQ at E , and BE intersect circumcircle of DPB and APC again at K and L respectively. Let $M = DK \cap AQ$. Given that $CL \parallel AB$, find $\frac{ME}{MK}$.

5. (Navi) Find the number of ordered quadruples (w, x, y, z) , where w, x, y, z are non-negative reals, that satisfy the equation

$$\left(\frac{1}{w} + \frac{1}{x} + \frac{1}{wx}\right)\left(\frac{1}{y} + \frac{1}{z} + \frac{1}{yz}\right) + 4 = \left(2 + \frac{1}{wxyz}\right)$$

6. (Melodies) Find the remainder when $(5 \cdot 15122014^{2014} + 1)(5 \cdot 15122014^{2014} + 2)(5 \cdot 15122014^{2014} + 3)(5 \cdot 15122014^{2014} + 4)$ is divided by 25.
7. (Guilherme) Let ψ be a complex number such that $(1 + 2i)^3 + (3 + 2i)^1 + \psi = 0$. Evaluate ψ 's real part.
8. (Guilherme) Consider the $\mathbb{N} \rightarrow \mathbb{R}_+$ function $f(x)$ such that the sum of the first n values of $f(x)$, starting at 1, is \sqrt{n} . Find the integer part I_p of the sum of the first 25 values of $\frac{1}{f(x)}$, starting at 1. As an example of nomenclature $I_p(2 + \sqrt{5}) = 2$, $I_p(7) = 7$, $I_p(\pi) = 0$.
9. (Guilherme) For $0 < x < 1$, let S_{red} be the area inside the red trapezoid, and S_{black} be the area between the black square and the red trapezoid. For $x = \frac{1}{\pi}$, find the sum of the first ten decimal places of the ratio between S_{red} and S_{black} , plus the integer part.



10. (Guilherme) P and Q be two-digit numbers such that the last three digits of P^2 and Q^2 are respectively 929 and 464. Evaluate the last three digits of PQ .

Long Questions

Explain your answer for each question.

1. A quadrilateral $ABCD$ has side lengths $AB = 5, BC = 6, CD = 8, DA = 7$. Prove that there exists a point P in $ABCD$ such that the perpendiculars from P to the sides of $ABCD$ are equal.
(2 Points)
2. (Navi) Do there exist positive integers x, y such that $\sqrt{5^x + 7^y}$ is an integer?
(2 points)
3. (ZS) Prove that $3^{2^n} - 1$ can be written as the sum of two squares for all positive integers n .
(3 points)