Competition #6

The Junior Online Math Olympiad

30th June 2014 - 7th July 2014

Short Questions

1. Sam has 143 diamonds. They are to be arranged into different boxes such that if Adi asks for any number of diamonds from 1 to 143, then Sam must be able to give them without opening any box (directly giving specific boxes). What is the minimum number of boxes Sam will need so that any number of diamonds can be given?

2. For how many positive integers less than 300 is $n^5 - n^2$ divisible by 25?

3. Given that a sequence starts with a_1 and that each term is defined by $a_n = n^3 - 4n - 1$. What are the last 3 digits of the sum of first 66 terms of this sequence ?

4. Adi likes to eat Pizza, he has 6 different Pizza bases , 7 flavours and 8 different toppings . But his doctor told him to avoid 1 of the pizza bases, 2 flavours and 1 type of topping, because it might be allergic for him. Then the ratio of Pizzas he can't eat to the Pizzas he can eat can be expressed as $\frac{a}{b}$ where $gcd(a, b) = 1$ find $a + b$

5. A man was standing on the top of a 300m tall tower on coast of arabic sea. He saw a ship in the sea which was coming towards the coast with a uniform velocity. The angle of depression with which he had to see the ship was 30°. After 40 seconds that angle of depression was seen to be 45°. Let the speed of the ship be x m/s. The ship needs y seconds more to reach the coast from the current point . Then the value of $x + y$ can be expressed as $\frac{a\sqrt{3}+b}{c}$ where c is a prime number . Find the value of $a+b+c$.

6. Find the number of real solutions of $\dfrac{x^4 - 20x^3 + 150x^2 - 500x + 625}{x - 5} = 0$

7. Given that

$$a+b+c=6$$
$$a^2+b^2+c^2=66$$
$$a^3+b^3+c^3=666$$

Then find the last 3 digits of $a^4+b^4+c^4$

8. Sam has a collection of numbers from 0 to 9, which has sum of all it's elements as '13'. If it's given that Adi *can* form a number divisible by 11 using all the digits of this collection exactly once, then which digit is surely in this set ?

9. Find the difference between the sum of the x-coordinates and the sum of y co-ordinates of the intersections of the curves $y=x^4+2x^3-21x^2-21x-40$ and $x=y^4+2y^3-21y^2-21y-40$

10. Find the sum of the values of x and y for all the positive integer solutions for the equation:

$$x^2-y^2=211$$

Long Questions

Explain your answer for each of the questions

1. Find, with proof, all values of $x \in \mathbb{N}$ such that $\frac{x^x+1}{x+1}$ is a natural number.

   (3 points)

2. Given that for $x,y,z \geq 0$, we have $xy+yz+zx=3$ Then prove that:

$$\frac{x+y+z}{x^2y^2z^2} \geq \frac{9}{x^3+y^3+z^3-(x+y+z)\left[(x+1)(x-1)+(y+1)(y-1)+(z-1)(z+1)\right]}$$

   (3 points)

3. A teacher writes down three numbers, 1, 2 and 3, on the whiteboard. Now, every student take turns to the whiteboard and erase one number, and then replace it by the sum of the two numbers left. After some turns, is it possible to have the numbers: $6^{2012}, 7^{2013}, 8^{2014}$ on the whiteboard at the same time? Give proof.

   (2 points)