Competition #7

The Junior Online Math Olympiad

28th July 2014 - 4th August 2014

Short Questions

1. How many rectangles are there in a 20×14 chess board?

2. If $(w-1)(w+1) = w$, find the value of $w^{10} + w^{-10}$.

3. What is the maximum number of regions in which 7 lines divide a plane into?

4. Trevor creates a infinite sequence S of integers by repeatedly adding the digits 2, 0, 1, and 3 to the end of the previous number. For example, the first numbers he creates are $2, 20, 201, 2013, 20132, 201320, 2013201\ldots$

   Find the number of primes in S.

5. Let the maximum value of $\frac{(x+y)^2}{x^2+y^2} = M$. Find M.

6. Yan Yau is going to school. There are 2 different buses that Yan Yau could take: the C4 and the C9. They both leave at the plaza and have different schedules. The C4 bus takes 9 minutes to get to school and the C9 takes 4 minutes.

   Here is a portion of the schedule for the C4 bus:

   $$10{:}10, 10{:}16, 10{:}28, 10{:}40, 10{:}46, 10{:}58, 11{:}10$$

   Here is a portion of the schedule for the C9 bus:

   $$10{:}04, 10{:}10, 10{:}22, 10{:}34, 10{:}40, 10{:}52, 11{:}04$$

   Yan Yau arrives at the bus stop at a random time between 10:00 and 11:00. He will take the first bus that leaves; if both buses depart at the same time, then he will choose the bus that takes the least time for him to get to school.

   The probability that Yan Yau takes the C9 bus can be expressed as $\frac{a}{b}$ where a and b are positive coprime integers. Find the value of $a + b$.

7. Given the same scenario as in Question #6, what is the expected number of minutes after 10 : 00 when Yan Yau arrives at school? Round your answer to the nearest integer.

8. Adi has a sequence of integers $\{a_i\}_{i=1}^4$ and Sam has a sequence of integers $\{b_i\}_{i=1}^4$, which are such that

$$1 \leq a_1 < a_2 < a_3 < a_4 \leq 20$$
$$1 \leq b_1 < b_2 < b_3 < b_4 \leq 20.$$

Their sequences are so cool that they satisfy the property

$$\frac{\sum_{i=1}^4 a_i}{\sum_{i=1}^4 b_i} = \frac{a_i}{b_i}$$

for all $i = 1, 2, 3, 4$. How many different pairs of sequences $(< a_i >, < b_i >)$ can they have?

9. Let I be the incenter of $\triangle ABC$, and let AI, BI, and CI intersect BC, CA, and AB at D, E, and F, respectively. If $AB = 20$, $BC = 14$ and $AC = \frac{438}{53}$, then $\dfrac{ID}{IF} = \dfrac{a}{b}$ for positive, co-prime integers a and b. Find the value of $a + b$.

10. Find the sum of all values of x that satisfy the equation

$$x^{x^5-37x^4+7x^3+84x^2+2x+9} = 1$$

Long Questions

Explain your answer for each question.

1. Find with proof, all integer values of k such that $2^k = k^2$.

   (2 points)

2. Ronald is playing around with arrow notation to denote exponentiation, where $a \uparrow b = a^b$. For example, $2 \uparrow 2 = 2^2 = 4$ and $3 \uparrow 4 = 3^4 = 81$.

   A "double arrow" is used for iterated exponentiation, as seen in the following:

$$a \uparrow\uparrow b = \underbrace{a^{a^{\cdots^{a}}}}_{b \text{ copies of } a}.$$

   For example,

$$2 \uparrow\uparrow 3 = 2^{2^2} = 2^4 = 16.$$

   You can use this information to extend the following.

$$a \uparrow\uparrow b = a^{a\uparrow\uparrow(b-1)}$$

Find, with proof, the last digit of $3 \uparrow\uparrow 2014$.

(2 points)

3. Find all roots, with proof, of the equation $x^4 - 22x^2 - 12x + 22 = 0$.

   (3 points)