Competition #8

The Junior Online Math Olympiad

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Short Questions

1. Euler's Phi function, written as $\varphi(x)$, is defined as the number of positive integers, smaller or equal to x, that are coprime to x. Its properties are:

   - if p is a random prime and q is a positive integer, then $\varphi(p^a) = p^a - p^{a-1}$;
   - if q and r are coprime integers, $\varphi(qr) = \varphi(q) \cdot \varphi(r)$;

   Evaluate the value of $\varphi(\varphi(1859) - \varphi(1573))$.

2. If the quintic polynomial $2x^5 - 7x^4 + 6x^3 - 21x^2 - 92x + 4$ has roots a, b, c, d, and e; find $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{e}$.

3. If given that the arithmetic mean of the squares of the numbers a and e is k then find the minimum value of

   $$(k+b^2+c^2+d^2+ab+bc+cd+de)\left(\frac{4a^2+5b^2+c^2+8ab+2bc}{(a+b)^2(b+c)^2}+\frac{9e^2+16c^2+25d^2+18de+32cd}{(c+d)^2(d+e)^2}\right)$$

4. Adi has bought a team in football league, and Sam wants to design a striped Tshirt of his team.Thee Tshirt can have maximum 5 different colors and a total of 9 vertical strips. If at all some color is given to adjacent strips, it will look like a broad strip, bad looking, so it's not done. To make it look in symmetry,the strips are arranged in such a way that if you start looking at the strips from left or from right, the pattern will look same. Find how many different possibilities are there for Adi's team's Tshirt.

5. Adi has a polynoial $p(x)$, and Sam has some polynomials by which we divide Adi's polynomial. When $p(x^6)$ is divided by Sam's polynomial $x+2$, the remainder obtained is π . The remainder when $p(x^6)$ is divided by Sam's other polynomial to $x^3 - 4x^2 + 8x - 8$ is R. Then the value of $\prod_{k=0}^{3} \cos\left(\frac{(2k+1)R}{18}\right)$ can be written as $\frac{a}{b}$ where a and b are coprime positive integers, find the value of $a+b$.

6. There are 3 people living in a house Aditya, Cody, Yan Yau and a dog.

   They cook some cookies in the morning so that they can eat in the day, whenever they want.

   The 3 people go out for work, first Aditya comes home and gives 1 cookie to dog. Then he eats $\frac{1}{3}$ of the remaining cookies and goes.

   Then Cody comes home, gives 1 cookie to dog and eats $\frac{1}{3}$ of what is remaining and goes to work again.

   Then Yan Yau comes home, gives 1 cookie to dog and eats $\frac{1}{3}$ of what is remaining, goes to work again.

   At night, all 3 of them come back home again, and then, they give one cookie to the dog and divide the remaining cookies in 3 equal shares and eat them.

   In this whole process, if no cookie was broken into parts, then the minimum number of cookies that was made by them in the morning is x.

   Number of cookies that Aditya ate is y. Find the value of $x+y$

7. The product of the first three terms of a five-term positive geometric progression is 9^6, and the product of the last three ones is 3^6. Evaluate the sum of all five terms.

8. Solve the system of equations $x+y=1, x^5+y^5=31$ and find the sum of all $|x_0|^2+|y_0|^2$ for all pairs of solutions (x, y)

9. Evaluate the sum of the squares of all integers that satisfy the equation:

$$\binom{36}{2x^2-x} = \binom{36}{9x+12}$$

10. Find the largest positive real number C such that for all positive real numbers a, b, c, d with sum 4:

$$\frac{a-1}{a^2+1}+\frac{b-1}{b^2+1}+\frac{c-1}{c^2+1}+\frac{d-1}{d^2+1}+C\sqrt[16]{(a^4+1)(b^4+1)(c^4+1)(d^4+1)} \leq 4$$

   $\log_2 C = \frac{x}{y}$ for coprime positive integers a and b, find the value of $x+y$

Long Questions

Explain your answer for each question.

1. Given that k is an irrational number, prove that $1 - k$ is also irrational.

   (2 points)

2. This question has been removed and omitted from marking, the word "non-trapezoid" was unnecessary in this question.

3. Given that $19 \mid 7x + 5y - 3z$ and $x, y, z \in \mathbb{Z}$. Prove that:

$$19 \mid 5x + 9y - 13z$$

   (2 Points)