Weekly Challenge 2015

The Junior Online Math Olympiad

Challenges

Week 3: (Michael Tang) Prove that there is no pair of real numbers (a, b) so that both of the systems

$$\begin{cases} x^2 + y^2 = a \\ xy = b \end{cases}$$

$$\begin{cases} x^2 + y^2 = b \\ xy = a \end{cases}$$

have positive real solutions for (x, y).

(3 Points)

Solution

For the first system, AM-GM leads us to $a > 2b$

For the second system, AM-GM leads us to $b > 2a$.

This means that $a > 4a$ and $b > 4b$, so a and b cannot be positive.

But $m^2 + n^2 > 0$, if (m, n) are real numbers.

Contradiction, hence proved.

Week 2: (Navi) Prove that for any positive integers $x \neq y$, there exist positive integers (m, n) such that:

$$\frac{x^4 + y^4 + m^4}{x^2 + y^2 + m^2} = m^2 + n$$

(2 Points)

Solution

Taking $m = |x - y|$ and $n = xy$, the equation above is always satisfied for any x, y.

Week 1: (ZS) Prove that $10a^2 + 9b^2$ and $10b^2 + 9a^2$ can't simultaneously be perfect squares for all positive integers a, b.

(3 points)

Solution

Suppose not, then $19(a^2 + b^2)$ can be written as the sum of two squares. But it's a well-known fact that the exponent of every prime congruent to 3 mod 4 is even if a number can be written as the sum of 2 squares. So, since the exponent of 19 of $19(a^2 + b^2)$ must be odd, it can't be written as the sum of two squares, a contradiction.